LECTURES

LESSON XII

12. Integral Equations

12.1. Introduction

In spite the fact that integral equations are almost never treated in numerical analysis textbooks, there is a large and growing literature on their numerical solution. One reason for the sheer volume of this activity is that there are many different kinds of equations, each with many different possible pitfalls. Often many different algorithms have been proposed to deal with a single case. There is a close correspondence between linear integral equations, which specify linear, integral relations among functions in an infinite-dimensional function space, and plain old linear equations, which specify analogous relations among vectors in a finite-dimensional vector space. This correspondence lies at the heart of most computational algorithms, as we in program realization of their numerical solution.

The equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)\,dt, \tag{12.1.1}$$

where f i K are known functions, y unknown function, and λ numerical parameter, is called a Fredholm integral equation of second kind. Fredholm equations involve definite integrals with fixed upper and lower limits.

The function in two variables K is called kernel of integral equation (12.1.1). In our considerations we will always suppose that kernel is defined and continuous on $D = \{(x,t)|a \le x \le b,\ a \le t \le b\}$.

If $f(x) \not\equiv 0$, the equation (12.1.1) is called inhomogeneous, and in case when $f(x) \equiv 0$, equation is homogenous.

Integral equation of form

$$f(x) + \lambda \int_a^b K(x,t)y(t)\,dt = 0$$

is called Fredholm integral equation of first kind. As we already noted, this equation can be written in analogous form, as matrix equation

$$\mathbf{K} \cdot \vec{y} = -\vec{f}$$

which solution is

$$\vec{y} = -\mathbf{K}^{-1} \cdot \vec{f},$$

and $\mathbf{K}^{-1}$ is matrix inverse. Both equation are solvable when function f and $\vec{f}$ are nonzero, respectively (the homogeneous case with f=0 is almost never useful), and $K(\mathbf{K})$ is invertible.

The analogous matrix form of Fredholm equation of second kind (12.1.1) is

$$(\mathbf{K} - \frac{1}{\lambda}\mathbf{I}) \cdot \vec{y} = -\frac{\vec{f}}{\lambda}.$$

Again, if f or $\vec{f}$ is zero, then the equation is said to be homogeneous. If the kernel $K(x,t)$ is bounded, then, like in matrix form, the equation (12.1.1) has the property that its homogeneous form has solutions for at most a denumerably infinite set $\lambda = \lambda_n,\ n = 1, 2 \ldots$, the eigenvalues. The corresponding solutions $y_n(x)$ are the eigenfunctions. The eigenvalues are real if the kernel is symmetric. In the inhomogeneous case of nonzero f or $\vec{f}$, both equations are solvable except when λ or $1/\lambda$ is an eigenvalue - because the integral operator (or matrix) is singular then. In integral equations this dichotomy is called the Fredholm alternative.

Fredholm equations of the first kind are often extremely ill-conditioned. Applying the kernel to a function is generally a smoothing operation, so the solution, which requires inverting the operator, will be extremely sensitive to small changes or errors in the input. Smoothing often actually loses information, and there is no way to get it back in an inverse operation. Specialized methods have been developed for such equations, which are often called inverse problems. The idea is that method must augment the information given with some prior knowledge of the nature of the solution. This prior knowledge is then used, in some way, to restore lost information.

Volterra integral equations of first and second kind are of forms

$$f(x) + \lambda \int_a^x K(x,t) y(t)\, dt = 0$$

and

$$y(x) = f(x) + \lambda \int_a^x K(x,t) y(t)\, dt\,,$$

respectively. Volterra equations are a special case of Fredholm equations with $K(x,t) = 0$ for $t > x$. Chopping off the unnecessary part of the integration, Volterra equations are written in a form where the upper limit of integration is the independent variable x. The analogous matrix form of Volterra equation of first kind (written out in components) is

$$\sum_{j=1}^{k} K_{kj} y_j = f_k,$$

wherefrom we see that Volterra equation corresponds to a matrix $\mathbf{K}$ that is lower (left) triangular. As we already know, such matrix equations are trivially soluble by forward substitution. Techniques for solving Volterra equations are similarly straightforward. When experimental measurement noise does not dominate, Volterra equations of the first kind tend not to be ill-conditioned. The upper limit to the integral introduces a sharp step that conveniently spoils any smoothing properties of the kernel. The matrix analog of Volterra equation of the second kind is

$$(\mathbf{K} - \mathbf{I}) \cdot \vec{y} = \vec{f},$$

with $\mathbf{K}$ lower triangular matrix. The reason there is no λ in these equations is that in inhomogeneous case (nonzero f) it can be absorbed into K or $\mathbf{K}$, while in the homogeneous case ($f = 0$), there is a theorem that Volterra equations of the second kind with bounded kernels have no eigenvalues with square-integrable eigenfunctions.

We have considered only the case of linear integral equations. The integrand in a nonlinear version of given equations of first kind (Fredholm and Volterra) would be $K(x,t,y(t))$ instead of $K(x,t)f(t)$, and a nonlinear versions of equations of second kind would have an integrand $K(x,t,y(x),y(t))$. Nonlinear Fredholm equations are considerably more complicated than their linear counterparts. Fortunately, they do not occur as frequently in practice. By contrast, solving nonlinear Volterra equations usually involves only a slight modification of the algorithm for linear equations. Almost all methods for solving integral equations numerically make use of quadrature rules, frequently Gaussian quadratures.

12.2. Method of successive approximations

For solving Fredholm equation (12.1.1) it is often used method of successive approximations based on equality

$$y_n(x) = f(x) + \lambda \int_a^b K(x,t)y_{n-1}(t)\,dt \quad (n = 1,2,\ldots), \tag{12.2.1}$$

whereby is taken $y_0 = f(x)$. Namely, if we define sequence of functions $\{\overline{y}_k\}$ by using

$$\overline{y}_0(x) = y_0(x) = f(x),\ \overline{y}_k(x) = \int_a^b K(x,t)\overline{y}_{k-1}(t)\,dt \quad (k = 1,2\ldots),$$

then (12.2.1) can be presented in the form

$$y_n(x) = \sum_{k=0}^{n} \lambda^k \overline{y}_k(x) \quad (n = 1,2,\ldots). \tag{12.2.2}$$

One can show that sequence y_n converges to exact solution of equation (12.1.1) if fulfilled the condition $|\lambda| < \dfrac{1}{M(b-a)}$ where

$$M = \max_{x,t\in[a,b]} |K(x,t)|$$

.

12.3. Application of quadrature formulas

In order to solve Fredholm equation (12.1.1) let's take quadrature formula

$$\int_a^b F(x)\,dx = \sum_{j=1}^{n} A_j F(x_j) + R_n(F), \tag{12.3.1}$$

where abscissas $x_1,\ldots,x_n$ are from $[a,b]$, A_j are weight coefficients not depending on F, and $R_n(F)$ corresponding residuum.

If we put in (12.1.1) successively $x = x_i$ $(i = 1,\ldots,n)$, we obtain

$$y(x_i) = f(x_i) + \lambda \int_a^b K(x,t)y(t)\,dt \quad (1 = 1,\ldots,n),$$

wherefrom by using quadrature formula (12.3.1) it follows

$$y(x_i) = f(x_i) + \lambda \sum_{k=1}^{n} A_j K(x_i,x_j)\,y(x_j) + R_n(F) \quad (i = 1,\ldots,n), \tag{12.3.2}$$

where $F_i(t) = K(x_i, t)\, y(t) \quad (i = 1, \ldots, n)$. By discarding members $R_n(F_i)$ $(i = 1, \ldots, n)$, based on (12.3.2) we get system of linear equations

$$y_i - \lambda \sum_{i=1}^{n} A_j K_{ij} y_j = f_i \quad (i = 1, \ldots, n), \tag{12.3.3}$$

where we put $y_i = y(x_i),\ f_i = f(x_i),\ K_{ij} = K(x_i, x_j)$. System (12.3.3) can also be given in matrix form

$$\begin{bmatrix} 1 - \lambda A_1 K_{11} & -\lambda A_2 K_{12} & \ldots & -\lambda A_n K_{1n} \\ -\lambda A_1 K_{21} & 1 - \lambda A_2 K_{22} & \ldots & -\lambda A_n K_{2n} \\ \vdots & & & \\ -\lambda A_1 K_{n1} & -\lambda A_2 K_{n2} & \ldots & 1 - \lambda A_n K_{nn} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

By solving the obtained system of linear equations in y_1,..., y_n, the approximative solution of equation (12.1.1) can be presented in the form

$$\tilde{y}(x) = f(x) + \lambda \sum_{j=1}^{n} A_j K(x, x_j)\, y_j. \tag{12.3.4}$$

12.4 Program realization

Method explained in previous section will be realized by using generalized Simpson quadrature formula, at which we have

$$h = \frac{b-a}{2m},\ n = 2m+1,\ x_i = a + (i-1)h \quad (i = 1, \ldots, n),$$

$$A_1 = A_{2m+1} = \frac{h}{3},\ A_2 = A_4 = \ldots = A_{2m} = \frac{4h}{3},$$

$$A_3 = A_5 = \ldots = A_{2m-1} = \frac{2h}{3}.$$

For solving system of linear equations (12.3.3) we will use subroutines `LRFAK` and `RSTS` The code of subroutines and description of subroutines parameters are given in Chapter 2.

In subroutine `FRED` is formed system of equations (12.3.3). Parameters in subroutine parameter list are of following meaning:

`X` - vector of abscissas of quadrature formula;

`A` - vector of weight coefficients of quadrature formula;

`FK` - name of function subroutine with function f and kernel K;

`PL` - parameter λ;

`C` - matrix of system (12.3.3), stored as vector in columnwise way (column by column);

`F` - vector of free members in system of equation (12.3.3).

Subroutine code is of form:

```
      SUBROUTINE FRED(X,A,N,FK,PL,C,F)
      DIMENSION X(1), A(1),C(1),F(1)
      IND=-N
      DO 15 J=1,N
      IND=IND+N
      DO 10 I=1,N
      IJ=IND+I
      C(IJ)=-PL*A(J)*FK(X(I),X(J),2)
      IF(I-J)10,5,10
5     C(IJ)=1+C(IJ)
10    CONTINUE
15    F(J)=FK(X(J),X(I),1)
```

```
      RETURN
      END
```

Function subroutine FK has a following parameters in the parameter list:

X and T - values of arguments x and t respectively.

M - integer which governs calculation of function f (M=1) and kernel K (M=2) for given values of arguments. Subroutine code is of form:

```
      FUNCTION FK(X,T,M)
      GO TO (10,20), M
10    FK=EXP(X)
      RETURN
 20   FK=X*EXP(X*T)
      RETURN
      END
```

Main program is organized in such a way that at first in FRED is formed system of equation, and then is matrix of system factorized by subroutine LRFAK, what enables solving of system of equations by subroutine **RSTS**.

Taking as an example equation

$$y(x) = e^x - \int_0^1 x e^{xt}\, y(t)\, dt$$

and M=1,2 (N=3,5), the corresponding results are obtained and given after main program code. Note that exact solution of given equation is $y(x) = 1$.

```
      EXTERNAL FK
      DIMENSION X(10), A(10), C(100),B(10),IP(9)
      OPEN(8,FILE='FRED.IN')
      OPEN(5,FILE='FRED.OUT')
      READ(8,5)PL,DG,GG
 5    FORMAT(3F5.0)
 10   READ(8,15,END=60) M
15    FORMAT(I2)
      N=2*M+1
      H=(GG-DG)/(2.*FLOAT(M))
      X(1)=DG
      DO 20 I=2,N
 20   X(I)=X(I-1)+H
      Q=H/3.
      A(1)=Q
      A(N)=Q
      DO 25 I=1,M
25    A(2*I)=4.*Q
      DO 30 I=2,M
30    A(2*I-1)=2.*Q
      CALL FRED(X,A,N,FK,PL,C,B)
      CALL LRFAK(C,N,IP,DET,KB)
      IF(KB) 35,40,35
 35   WRITE(5,45)
 45   FORMAT(1H0,'MATRICA SISTEMA SINGULARNA'//)
      GO TO 60
40    CALL RSTS(C,N,IP,B)
      WRITE(5,50)(B(I),I=1,N)
 50   FORMAT(/5X,'RESENJE'//(10F10.5))
      GO TO 10
60    CLOSE(5)
      CLOSE(8)
      STOP
      END
```

```
     RESENJE
```

```
1.00000   0.94328   0.79472
  RESENJE
1.00000   1.00000   1.00000   1.00000   0.99998
```

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